Emission Coordinates

```
x = {1, 0, 0};
y = {0, 1, 0};
z = {0, 0, 1};
(*
Rotation order: θ → β → χ
*)
(*
  y, x (rot about z)
  z, y (rot about x)
  x, z (rot about y)
*)
Clear[α, β, θ, χ, ψ, ϕ, k, ν, ξ, k, kk, kx, ky, kz];
Mchi = RotationMatrix[-χ, {x, y}];
Mbeta = RotationMatrix[β, {y, z}];
Mtheta = RotationMatrix[-θ, {z, x}];
(*
     To get the velocity off the sample face in the sample coordinate frame,
we need to perform passive rotations of the
     analyzer coordinate frame back to that of the sample. Throughout
 we use the PyARPES angular conventions. We also assume no
     angular offsets for the sample, i.e. we assume that the
 sample is glued perfectly to the face of the sample puck. This is
     almost but not quite identical to just choosing offsets in
 software that place Γ at the appropriate coordinates, and is valid
     so long as the offset angles between sample and puck normal are
 small angles (you can check this by working through the full coordinate
     transforms and applying the small angle approximation.
   *)
Vemissionspherical = {Cos[ν] Sin[ξ], Sin[ν] Sin[ξ], Cos[ξ]};
(* Spherical coordinates with declination angle ξ and azimuthal angle ν. *)
(* Vemission = Mchi.Mbeta.Mtheta.Vanaframe; *)
MatrixForm[Vemissionspherical]
MatrixForm[Mchi.Mbeta.Mtheta]
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\nu]\ \mathrm{Sin}[\xi] \\ \mathrm{Sin}[\xi]\ \mathrm{Sin}[\nu] \\ \mathrm{Cos}[\xi] \end{pmatrix}$$

Out[13]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\theta]\ \mathrm{Cos}[\chi] - \mathrm{Sin}[\beta]\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\chi] & \mathrm{Cos}[\beta]\ \mathrm{Sin}[\chi] & -\mathrm{Cos}[\chi]\ \mathrm{Sin}[\theta] - \mathrm{Cos}[\theta]\ \mathrm{Sin}[\beta]\ \mathrm{Sin}[\\ -\mathrm{Cos}[\chi]\ \mathrm{Sin}[\beta]\ \mathrm{Sin}[\theta] - \mathrm{Cos}[\theta]\ \mathrm{Sin}[\chi] & \mathrm{Cos}[\beta]\ \mathrm{Cos}[\chi] & -\mathrm{Cos}[\theta]\ \mathrm{Cos}[\chi]\ \mathrm{Sin}[\beta] + \mathrm{Sin}[\theta]\ \mathrm{Sin}[\\ \mathrm{Cos}[\beta]\ \mathrm{Sin}[\theta] & \mathrm{Sin}[\beta] & \mathrm{Cos}[\beta]\ \mathrm{Cos}[\theta] \end{pmatrix}$$

Analyzer's Coordinates

The analyzer measures photoelectron velocities and records them in terms of three angles: ϕ (along the slit), ψ (electrostatic slit deflector angle or analyzer motion angle) and α (analyzer rotation). If temporarily we assume α == 0 then

In[14]:=
```
Clear[α, ϕ, ψ]
Vanaframe = EulerMatrix[{α, ϕ, -ψ}, {3, 2, 1}];
MatrixForm[Vanaframe]
MatrixForm[Vanaframe.z]
Assuming[
 {α == 0},
 MatrixForm[Simplify[Vanaframe.z]]
]
Assuming[
 {α == π/2},
 MatrixForm[Simplify[Vanaframe.z]]
]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} \cos[\alpha]\cos[\phi] & -\cos[\psi]\sin[\alpha]-\cos[\alpha]\sin[\phi]\sin[\psi] & \cos[\alpha]\cos[\psi]\sin[\phi]-\sin[\alpha]\sin[\psi] \\ \cos[\phi]\sin[\alpha] & \cos[\alpha]\cos[\psi]-\sin[\alpha]\sin[\phi]\sin[\psi] & \cos[\psi]\sin[\alpha]\sin[\phi]+\cos[\alpha]\sin[\psi] \\ -\sin[\phi] & -\cos[\phi]\sin[\psi] & \cos[\phi]\cos[\psi] \end{pmatrix}$$

Out[17]//MatrixForm=

$$\begin{pmatrix} \cos[\alpha]\cos[\psi]\sin[\phi]-\sin[\alpha]\sin[\psi] \\ \cos[\psi]\sin[\alpha]\sin[\phi]+\cos[\alpha]\sin[\psi] \\ \cos[\phi]\cos[\psi] \end{pmatrix}$$

Out[18]//MatrixForm=

$$\begin{pmatrix} \cos[\psi]\sin[\phi] \\ \sin[\psi] \\ \cos[\phi]\cos[\psi] \end{pmatrix}$$

Out[19]//MatrixForm=

$$\begin{pmatrix} -\sin[\psi] \\ \cos[\psi]\sin[\phi] \\ \cos[\phi]\cos[\psi] \end{pmatrix}$$

Momentum Conservation

We know that the kinetic energy and momentum are related to the velocity and emission angles by

```
In[20]:= kc = Sqrt[2 * m * Ek] / ℏ
         k = kc * Vemissionspherical;
         MatrixForm[k]
```

Out[20]= $\frac{\sqrt{2}\,\sqrt{\text{Ek m}}}{\hbar}$

Out[22]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{2}\,\sqrt{\text{Ek m}}\,\text{Cos}[\nu]\,\text{Sin}[\zeta]}{\hbar} \\ \frac{\sqrt{2}\,\sqrt{\text{Ek m}}\,\text{Sin}[\zeta]\,\text{Sin}[\nu]}{\hbar} \\ \frac{\sqrt{2}\,\sqrt{\text{Ek m}}\,\text{Cos}[\zeta]}{\hbar} \end{pmatrix}$$

Inverting for the analyzer angles in terms of k

We now have a set of three equations coupling analyzer and scan angles to the momentum. In principle now, we can calculate the momentum directly from the experimental degrees of freedom. However, if our data is gridded, we need to convert backwards from momentum to angles in order to be able to interpolate. To do this, we will see a few special cases below.

```
In[23]:= momentum = kc * MatrixForm[(Mchi.Mbeta.Mtheta).(Vanaframe.z)]
         TransAna = kk * Mchi.Mbeta.Mtheta.(Vanaframe.z);
         MatrixForm[TransAna]
```

Out[23]= $\frac{1}{\hbar}\sqrt{2}\,\sqrt{\text{Ek m}} \begin{pmatrix} \text{Cos}[\phi]\,\text{Cos}[\psi]\,(-\text{Cos}[\chi]\,\text{Sin}[\theta] - \text{Cos}[\theta]\,\text{Sin}[\beta]\,\text{Sin}[\chi]) + \text{Cos}[\beta]\,\text{Sin}[\chi]\,(\text{Cos}[\psi \\ \text{Cos}[\phi]\,\text{Cos}[\psi]\,(-\text{Cos}[\theta]\,\text{Cos}[\chi]\,\text{Sin}[\beta] + \text{Sin}[\theta]\,\text{Sin}[\chi]) + \text{Cos}[\beta]\,\text{Cos}[\chi]\,(\text{Cos}[\psi] \\ \text{Cos}[\beta]\,\text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] + \text{Sin}[\beta]\,(\text{Cos}[\psi]\,\text{Sir} \end{pmatrix}$

Out[25]//MatrixForm=

$$\begin{pmatrix} \text{kk}\,(\text{Cos}[\phi]\,\text{Cos}[\psi]\,(-\text{Cos}[\chi]\,\text{Sin}[\theta] - \text{Cos}[\theta]\,\text{Sin}[\beta]\,\text{Sin}[\chi]) + \text{Cos}[\beta]\,\text{Sin}[\chi]\,(\text{Cos}[\psi]\,\text{Sin}[\alpha] \\ \text{kk}\,(\text{Cos}[\phi]\,\text{Cos}[\psi]\,(-\text{Cos}[\theta]\,\text{Cos}[\chi]\,\text{Sin}[\beta] + \text{Sin}[\theta]\,\text{Sin}[\chi]) + \text{Cos}[\beta]\,\text{Cos}[\chi]\,(\text{Cos}[\psi]\,\text{Sin}[\alpha]\,\text{S} \\ \text{kk}\,(\text{Cos}[\beta]\,\text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] + \text{Sin}[\beta]\,(\text{Cos}[\psi]\,\text{Sin}[\alpha]\,\text{Sin}[\end{pmatrix}$$

Inverting a very simple case:

Fixed hemispherical analyzer with no sample or analyzer rotation

In this case, we can assume $\{\alpha,\chi,\psi\}$ == 0. This gives a very simple relationship between the sample angles, a single analyzer angle ϕ and the photoelectron momentum for horizontal slit analyzers.

In[26]:=
```
Assuming[
 {α == 0, χ == 0, ψ == 0},
 MatrixForm[{kx, ky, kz}] == MatrixForm[Simplify[TransAna]]
]
```

Out[26]= $\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,\mathrm{Sin}[\theta-\phi] \\ -kk\,\mathrm{Cos}[\theta-\phi]\,\mathrm{Sin}[\beta] \\ kk\,\mathrm{Cos}[\beta]\,\mathrm{Cos}[\theta-\phi] \end{pmatrix}$

Using the kx and ky equations we can invert for the “scan coordinate” β and the analyzer coordinate ϕ:

In[27]:=
```
Assuming[
 {α == 0, χ == 0, ψ == 0},
 FullSimplify[Solve[{(Simplify[TransAna].x /. (θ - ϕ) → δ) == kx}, {δ}]]
]
Assuming[
 {α == 0, χ == 0, ψ == 0},
 FullSimplify[Solve[{(Simplify[TransAna].y) == ky}, {β}]]
]
```

Out[27]= $\left\{\left\{\delta \to \mathrm{ConditionalExpression}\left[-\mathrm{ArcSin}\left[\frac{kx}{kk}\right]+2\pi c_1,\ c_1\in\mathbb{Z}\right]\right\},\right.$
$\left.\left\{\delta \to \mathrm{ConditionalExpression}\left[\pi+\mathrm{ArcSin}\left[\frac{kx}{kk}\right]+2\pi c_1,\ c_1\in\mathbb{Z}\right]\right\}\right\}$

Out[28]= $\left\{\left\{\beta \to \mathrm{ConditionalExpression}\left[-\mathrm{ArcSin}\left[\frac{ky\,\mathrm{Sec}[\theta-\phi]}{kk}\right]+2\pi c_1,\ c_1\in\mathbb{Z}\right]\right\},\right.$
$\left.\left\{\beta \to \mathrm{ConditionalExpression}\left[\pi+\mathrm{ArcSin}\left[\frac{ky\,\mathrm{Sec}[\theta-\phi]}{kk}\right]+2\pi c_1,\ c_1\in\mathbb{Z}\right]\right\}\right\}$

We can now plot what this looks like in terms of kx, ky:

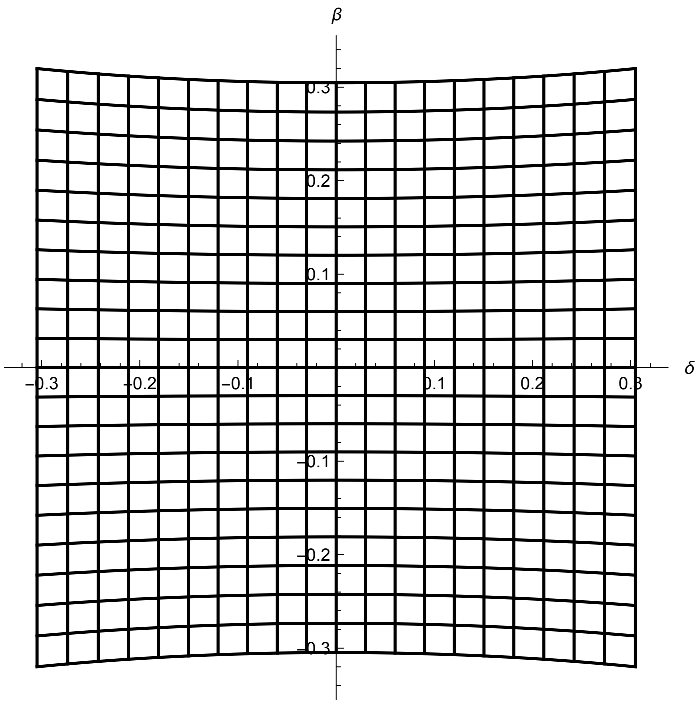
In[148]:= $\mathrm{FullSimplify}\left[-\mathrm{ArcSin}\left[\frac{ky\,\mathrm{Sec}[\theta-\phi]}{kk}\right]\ /.\ \theta \to \left(\phi-\mathrm{ArcSin}\left[\frac{kx}{kk}\right]\right)\right]$

Out[148]= $-\mathrm{ArcSin}\left[\frac{ky}{kk\sqrt{1-\frac{kx^2}{kk^2}}}\right]$

In[•]:=

```
kk = 1;
Show[ParametricPlot[
  Evaluate[Table[Tooltip[{-ArcSin[kx/kk], -ArcSin[(ky Sec[-ArcSin[kx/kk]])/kk]},
     Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]], {ky, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {δ, β}], ParametricPlot[
  Evaluate[Table[Tooltip[{-ArcSin[kx/kk], -ArcSin[(ky Sec[-ArcSin[kx/kk]])/kk]},
     Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]], {kx, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {δ, β}]]
```

Out[•]=



$\alpha = 0$ and $\chi \neq 0$

The general horizontal slit

Here, the simplest thing to do is to rotate the momentum coordinates by -χ and then apply our prior solution.

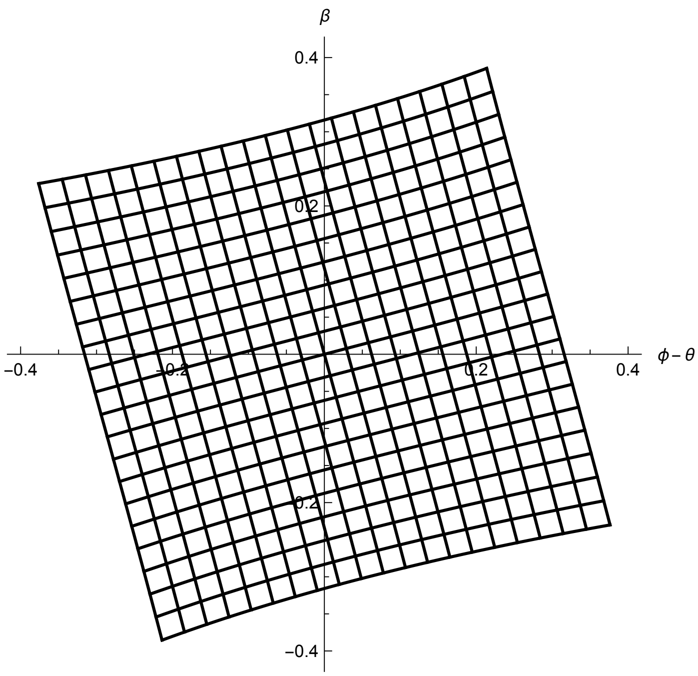
In[•]:=
```
Assuming[
 {α == 0, ψ == 0},
 MatrixForm[ RotationMatrix[χ, {x, y}].{kx, ky, kz}] ==
  MatrixForm[Simplify[RotationMatrix[χ, {x, y}].Simplify[TransAna]]]
]
```

Out[•]=

$$\begin{pmatrix} kx\,Cos[\chi] - ky\,Sin[\chi] \\ ky\,Cos[\chi] + kx\,Sin[\chi] \\ kz \end{pmatrix} == \begin{pmatrix} -Sin[\theta-\phi] \\ -Cos[\theta-\phi]\,Sin[\beta] \\ Cos[\beta]\,Cos[\theta-\phi] \end{pmatrix}$$

In[•]:=
```
kk = 1; χ = π / 12;
Show[ParametricPlot[Evaluate[Table[Tooltip[{-ArcSin[(kx Cos[χ] - ky Sin[χ])/kk],
     -ArcSin[((ky Cos[χ] + kx Sin[χ]) Sec[-ArcSin[(kx Cos[χ] - ky Sin[χ])/kk]])/kk]},
     Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]], {ky, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ - θ, β}],
 ParametricPlot[Evaluate[Table[Tooltip[{-ArcSin[(kx Cos[χ] - ky Sin[χ])/kk],
     -ArcSin[((ky Cos[χ] + kx Sin[χ]) Sec[-ArcSin[(kx Cos[χ] - ky Sin[χ])/kk]])/kk]},
     Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]], {kx, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ - θ, β}]]
```

Out[•]=



$\alpha = \pi/2$ and $\chi = 0$

In[•]:=
```
kk = 1; β = 0;
Show[ParametricPlot[Evaluate[
   Table[Tooltip[{-ArcSin[(kx Sqrt[1 + (kk^2+kx^2-2 Sqrt[kk^2-kx^2-ky^2] Abs[ky] Sin[2 β]+(kk^2-kx^2-2 ky^2) Cos[2 β])/(2 kk^2)])/kk],
        Sign[ky] * (π / 4 - ArcTan[
             Sqrt[kk^2 + kx^2 - 2 Sqrt[kk^2 - kx^2 - ky^2] Abs[ky] Sin[2 β] + (kk^2 - kx^2 - 2 ky^2) Cos[2 β]]/(Sqrt[2] kk)])}
        , Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]],
  {ky, -0.3, 0.3}, PlotStyle → {Black}, AspectRatio → 1,
  AxesLabel → {θ, ϕ}], ParametricPlot[Evaluate[
   Table[Tooltip[{-ArcSin[(kx Sqrt[1 + (kk^2+kx^2-2 Sqrt[kk^2-kx^2-ky^2] Abs[ky] Sin[2 β]+(kk^2-kx^2-2 ky^2) Cos[2 β])/(2 kk^2)])/kk],
        Sign[ky] (π / 4 - ArcTan[
             Sqrt[kk^2 + kx^2 - 2 Sqrt[kk^2 - kx^2 - ky^2] Abs[ky] Sin[2 β] + (kk^2 - kx^2 - 2 ky^2) Cos[2 β]]/(Sqrt[2] kk)])}
        , Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]], {kx, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {θ, ϕ}]]
Clear[
  kk,
  β];
```

$\alpha = \pi/2$ and $\chi = 0$

Small angle approximation version

In[ ]:=
```
Assuming[
 {α == π / 2, χ == 0, ψ == 0, β == 0},
 MatrixForm[{kx, ky, kz}] == MatrixForm[FullSimplify[TransAna] /. ϕ → ϕ - β]
]
```

Out[ ]=
$$\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,\mathrm{Cos}[\beta - \phi]\,\mathrm{Sin}[\theta] \\ -kk\,\mathrm{Sin}[\beta - \phi] \\ kk\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\beta - \phi] \end{pmatrix}$$

In[ ]:=
```
Assuming[
 {α == π / 2, χ == 0, ψ == 0},
 FullSimplify[Solve[{(-kk Sin[β - ϕ]) == ky}, {ϕ}]]
]
```

Out[ ]=
$$\left\{\left\{\phi \to \mathrm{ConditionalExpression}\left[\beta + \mathrm{ArcCos}\left[\frac{ky}{kk}\right] - \frac{1}{2}\pi\left(3 + 4\,c_1\right),\ c_1 \in \mathbb{Z}\right]\right\},\right.$$
$$\left.\left\{\phi \to \mathrm{ConditionalExpression}\left[\beta + \mathrm{ArcSin}\left[\frac{ky}{kk}\right] - 2\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

In[∘]:=
```
Assuming[
  {α == π/2, χ == 0, ψ == 0},
  FullSimplify[Solve[{(-kk Cos[β - ϕ] Sin[θ] /. ϕ → (β + ArcSin[ky/kk])) == kx}, {θ}]]
]
```

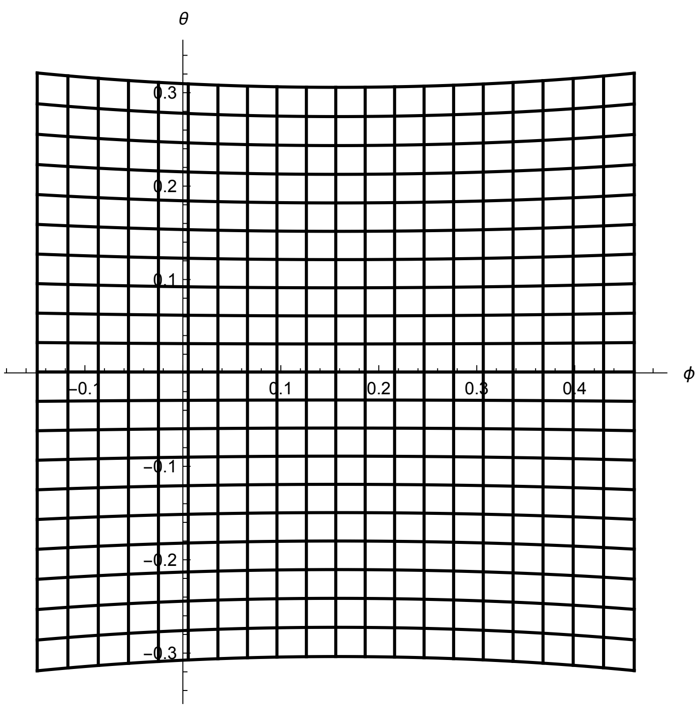
Out[∘]=

$$\left\{\left\{\theta \to \text{ConditionalExpression}\left[-\text{ArcSin}\left[\frac{kx}{kk\sqrt{1-\frac{ky^2}{kk^2}}}\right]+2\,\pi\,c_1,\ c_1\in\mathbb{Z}\right]\right\},\right.$$

$$\left.\left\{\theta \to \text{ConditionalExpression}\left[\pi+\text{ArcSin}\left[\frac{kx}{kk\sqrt{1-\frac{ky^2}{kk^2}}}\right]+2\,\pi\,c_1,\ c_1\in\mathbb{Z}\right]\right\}\right\}$$

In[◦]:=
```
kk = 1; β = π / 20;
Show[ParametricPlot[Evaluate[
    Table[Tooltip[{β + ArcSin[ky/kk], -ArcSin[kx/(kk Sqrt[1 - ky^2/kk^2])]}, Row[{"kx = ", kx}]],
     {kx, -0.301, 0.299, 0.03}]], {ky, -0.301, 0.299},
   PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, θ}],
  ParametricPlot[Evaluate[Table[Tooltip[{β + ArcSin[ky/kk], -ArcSin[kx/(kk Sqrt[1 - ky^2/kk^2])]},
      Row[{"ky = ", ky}]], {ky, -0.301, 0.299, 0.03}]], {kx, -0.301, 0.299},
   PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, θ}]]
Clear[
  kk,
  β];
```

Out[◦]=



$\alpha = \pi/2$ and $\chi \neq 0$

Small angle approximation for vertical slits and sample rotation

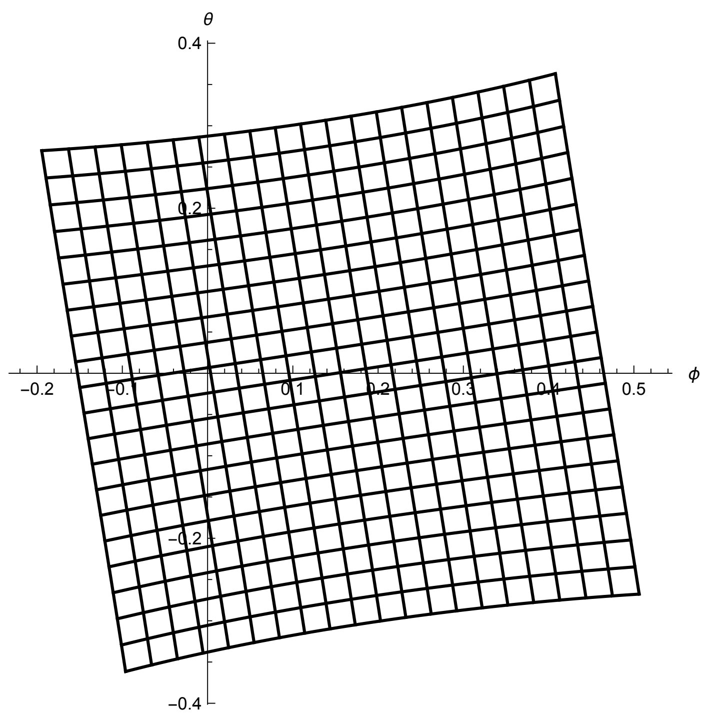
In[29]:=
```
Assuming[
 {α == 0, ψ == 0},
 MatrixForm[ RotationMatrix[χ, {x, y}].{kx, ky, kz}] == MatrixForm[
   Simplify[RotationMatrix[χ, {x, y}].(FullSimplify[TransAna] /. ϕ → ϕ - β)]]
 ]
```

Out[29]=

$$\begin{pmatrix} kx\,Cos[\chi] - ky\,Sin[\chi] \\ ky\,Cos[\chi] + kx\,Sin[\chi] \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,Sin[\beta + \Theta - \phi] \\ -kk\,Cos[\beta + \Theta - \phi]\,Sin[\beta] \\ kk\,Cos[\beta]\,Cos[\beta + \Theta - \phi] \end{pmatrix}$$

In[99]:=
```
kk = 1; β = π / 20; χ = π / 20;
Show[ParametricPlot[Evaluate[Table[
    Tooltip[{β + ArcSin[(ky Cos[χ] + kx Sin[χ])/kk], -ArcSin[(kx Cos[χ] - ky Sin[χ])/(kk Sqrt[1 - (ky Cos[χ]+kx Sin[χ])^2/kk^2])]},
     Row[{"kx = ", kx}]], {kx, -0.301, 0.299, 0.03}]],
   {ky, -0.301, 0.299}, PlotStyle → {Black}, AspectRatio → 1,
   AxesLabel → {ϕ, θ}], ParametricPlot[Evaluate[Table[
    Tooltip[{β + ArcSin[(ky Cos[χ] + kx Sin[χ])/kk], -ArcSin[(kx Cos[χ] - ky Sin[χ])/(kk Sqrt[1 - (ky Cos[χ]+kx Sin[χ])^2/kk^2])]},
     Row[{"ky = ", ky}]], {ky, -0.301, 0.299, 0.03}]], {kx, -0.301, 0.299},
   PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, θ}]]
Clear[
  kk,
  β];
```

Out[100]=



$\alpha = 0, \chi = 0, \psi \neq 0$

ARToF and Scanning Deflector Hemispheres, Small Angle Approximation

In[75]:=
```
Clear[α, ϕ, ψ, χ, θ, β];
Assuming[
    {α == 0, χ == 0, θ == 0},
    (* ϕ here refers to ϕ-θ *)
    MatrixForm[ {kx, ky, kz}] == MatrixForm[FullSimplify[TransAna]]
]
```

Out[76]=
$$\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} kk\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] \\ kk\,\left(-\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\beta] + \mathrm{Cos}[\beta]\,\mathrm{Sin}[\psi]\right) \\ kk\,\left(\mathrm{Cos}[\beta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] + \mathrm{Sin}[\beta]\,\mathrm{Sin}[\psi]\right) \end{pmatrix}$$

```
Assuming[
    {α == 0, χ == 0, β == 0},
    MatrixForm[{kx, ky, kz}] == MatrixForm[FullSimplify[TransAna] /. ψ → ψ - β]
]
```

Out[68]=
$$\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,\mathrm{Cos}[\beta-\psi]\,\mathrm{Sin}[\theta-\phi] \\ -kk\,\mathrm{Sin}[\beta-\psi] \\ kk\,\mathrm{Cos}[\theta-\phi]\,\mathrm{Cos}[\beta-\psi] \end{pmatrix}$$

In[80]:=
```
Assuming[
    {α == 0,  χ == 0, β == 0,  θ == 0},
    Solve[{-kk Sin[β - ψ] == ky}, {ψ}]
]
```

Out[80]=
$$\left\{\left\{\psi \to \mathrm{ConditionalExpression}\left[-\pi + \beta - \mathrm{ArcSin}\left[\frac{ky}{kk}\right] - 2\,\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\},\right.$$
$$\left.\left\{\psi \to \mathrm{ConditionalExpression}\left[\beta + \mathrm{ArcSin}\left[\frac{ky}{kk}\right] - 2\,\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

In[81]:=
```
FullSimplify[-kk Cos[β - ψ] Sin[θ - ϕ] /. ψ → β + ArcSin[ky/kk]]
```

Out[81]=
$$-kk\,\sqrt{1 - \frac{ky^2}{kk^2}}\;\mathrm{Sin}[\theta - \phi]$$

In[82]:=
```
Assuming[
    {α == 0,  χ == 0, β == 0,  θ == 0},
    Solve[{-kk Sqrt[1 - ky^2/kk^2] Sin[θ - ϕ] == kx}, {ϕ}]
]
```

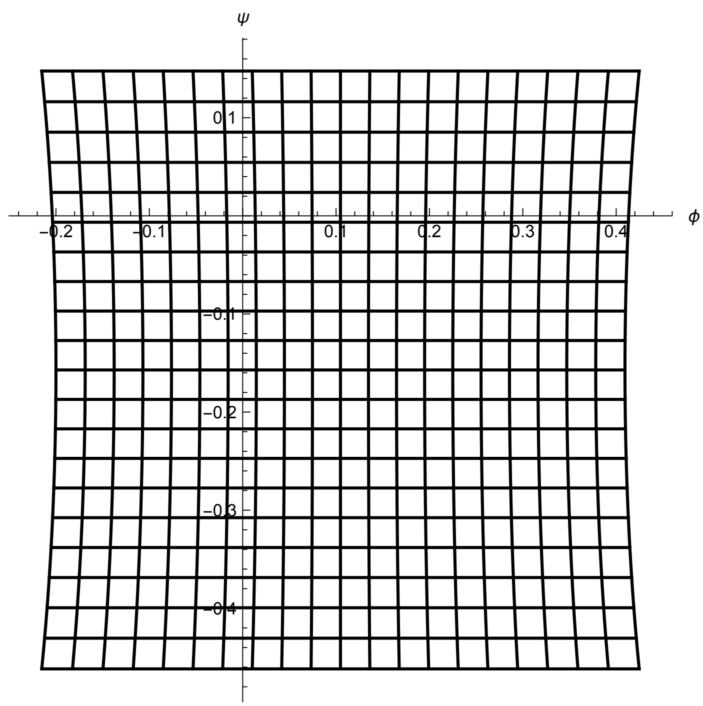
Out[82]=

$$\left\{\left\{\phi \to \text{ConditionalExpression}\left[-\pi + \theta - \text{ArcSin}\left[\frac{kx}{kk\sqrt{\frac{kk^2-ky^2}{kk^2}}}\right] - 2\pi c_1,\ c_1 \in \mathbb{Z}\right]\right\},\right.$$

$$\left.\left\{\phi \to \text{ConditionalExpression}\left[\theta + \text{ArcSin}\left[\frac{kx}{kk\sqrt{\frac{kk^2-ky^2}{kk^2}}}\right] - 2\pi c_1,\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

In[132]:=
```
kk = 1; β = -π/20;  θ = π/30; χ = 0;
Show[
 ParametricPlot[Evaluate[Table[Tooltip[{θ + ArcSin[kx/(kk Sqrt[(kk^2-ky^2)/kk^2])], β + ArcSin[ky/kk]},
     Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]], {ky, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}],
 ParametricPlot[Evaluate[Table[Tooltip[{θ + ArcSin[kx/(kk Sqrt[(kk^2-ky^2)/kk^2])], β + ArcSin[ky/kk]},
     Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]], {kx, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}]]
Clear[kk, β, θ, χ];
```

Out[133]=



In[98]:=

$\alpha = 0, \chi \neq 0, \psi \neq 0$

ARToF and Scanning Deflector Hemispheres, General case

In[137]:=
```
Clear[α, ϕ, ψ, χ, θ, β];
Assuming[
    {α == 0, χ == 0, θ == 0},
    (* ϕ here refers to ϕ-θ *)
    MatrixForm[RotationMatrix[χ, {x, y}].{kx, ky, kz}] ==
  MatrixForm[RotationMatrix[χ, {x, y}].FullSimplify[TransAna]]
]
```

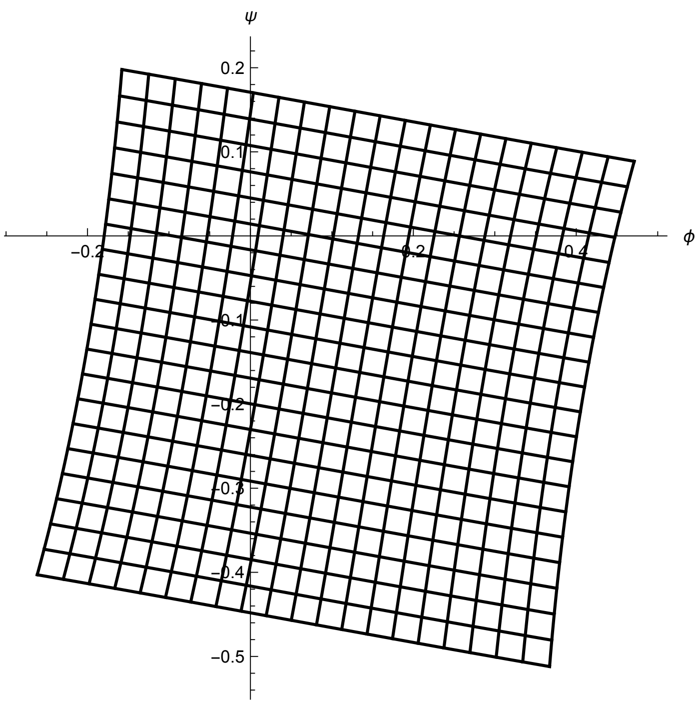
Out[138]=

$$\begin{pmatrix} kx\,Cos[\chi] - ky\,Sin[\chi] \\ ky\,Cos[\chi] + kx\,Sin[\chi] \\ kz \end{pmatrix} ==$$

$$\begin{pmatrix} kk\,Cos[\chi]\,Cos[\psi]\,Sin[\phi] - kk\,Sin[\chi]\,(-Cos[\phi]\,Cos[\psi]\,Sin[\beta] + Cos[\beta]\,Sin[\psi]) \\ kk\,Cos[\psi]\,Sin[\phi]\,Sin[\chi] + kk\,Cos[\chi]\,(-Cos[\phi]\,Cos[\psi]\,Sin[\beta] + Cos[\beta]\,Sin[\psi]) \\ kk\,(Cos[\beta]\,Cos[\phi]\,Cos[\psi] + Sin[\beta]\,Sin[\psi]) \end{pmatrix}$$

In[145]:=
```
kk = 1; β = -π/20; θ = π/30; χ = -π/18;
Show[ParametricPlot[Evaluate[Table[
    Tooltip[{θ + ArcSin[(kx Cos[χ] - ky Sin[χ]) / (kk Sqrt[(kk^2 - (ky Cos[χ] + kx Sin[χ])^2) / kk^2])], β + ArcSin[(ky Cos[χ] + kx Sin[χ]) / kk]},
     Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]], {ky, -0.3, 0.3},
  PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}],
 ParametricPlot[Evaluate[Table[Tooltip[{θ + ArcSin[(kx Cos[χ] - ky Sin[χ]) / (kk Sqrt[(kk^2 - (ky Cos[χ] + kx Sin[χ])^2) / kk^2])],
      β + ArcSin[(ky Cos[χ] + kx Sin[χ]) / kk]}, Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]],
  {kx, -0.3, 0.3}, PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}]]
Clear[kk, β, θ, χ];
```

Out[146]=



$\alpha = \pi/2, \chi = 0, \psi \neq 0$

ARToFs and Scanning Slit Hemispheres

In[224]:=
```
Clear[α, ϕ, ψ, χ, θ, β, kk];
Assuming[
    {α == π/2, χ == 0, θ == 0},
    MatrixForm[ {kx, ky, kz}] == Simplify[MatrixForm[TransAna]]
]
```

Out[225]=
$$\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,\mathrm{Sin}[\psi] \\ -kk\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\beta-\phi] \\ kk\,\mathrm{Cos}[\beta-\phi]\,\mathrm{Cos}[\psi] \end{pmatrix}$$

$\psi + \theta$ and $\beta - \psi$

In[220]:=
```
Clear[α, ϕ, ψ, χ, θ, β, kk];
Assuming[
    {α == π/2, χ == 0, β == 0, θ == 0},
    MatrixForm[ {kx, ky, kz}] ==
  TrigExpand[Simplify[MatrixForm[TransAna]] /. ψ → ψ + θ]
]
```

Out[221]=
$$\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} == \begin{pmatrix} -kk\,\mathrm{Sin}[\theta+\psi] \\ kk\,\mathrm{Cos}[\theta+\psi]\,\mathrm{Sin}[\phi] \\ kk\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\theta+\psi] \end{pmatrix}$$

In[227]:=
```
Assuming[
    {α == π/2, χ == 0, θ == 0},
        Solve[FullSimplify[TransAna].x == kx,  {ψ}]
]
```

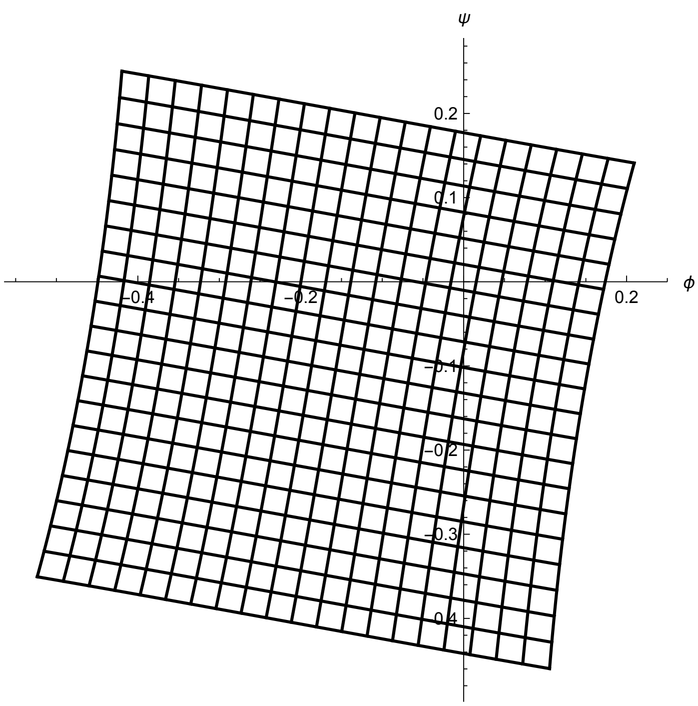
Out[227]=
$$\left\{\left\{\psi \to \mathrm{ConditionalExpression}\left[-\mathrm{ArcSin}\left[\frac{kx}{kk}\right] + 2\,\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\},\ \left\{\psi \to \mathrm{ConditionalExpression}\left[\pi + \mathrm{ArcSin}\left[\frac{kx}{kk}\right] + 2\,\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

In[228]:=
```
Assuming[
    {α == π/2, χ == 0, θ == 0},
    FullSimplify[
  Solve[FullSimplify[(FullSimplify[TransAna].y) /. ψ → (-ArcSin[kx/kk])] == ky, {ϕ}]]
]
```

Out[228]=
$$\left\{\left\{\phi \to \mathrm{ConditionalExpression}\left[\beta + \mathrm{ArcCos}\left[\frac{ky}{kk\sqrt{1-\frac{kx^2}{kk^2}}}\right] - \frac{1}{2}\,\pi\,\left(3 + 4\,c_1\right),\ c_1 \in \mathbb{Z}\right]\right\},\ \left\{\phi \to \mathrm{ConditionalExpression}\left[\beta + \mathrm{ArcSin}\left[\frac{ky}{kk\sqrt{1-\frac{kx^2}{kk^2}}}\right] - 2\,\pi\,c_1,\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

In[229]:=
```
kk = 1; β = -π/20;  θ = π/30; χ = -π/18;
Show[
 ParametricPlot[Evaluate[Table[Tooltip[{β + ArcSin[(ky Cos[χ] + kx Sin[χ])/(kk Sqrt[1 - (kx Cos[χ] - ky Sin[χ])^2/kk^2])], -θ -
       ArcSin[(kx Cos[χ] - ky Sin[χ])/kk]}, Row[{"kx = ", kx}]], {kx, -0.3, 0.3, 0.03}]],
  {ky, -0.3, 0.3}, PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}],
 ParametricPlot[Evaluate[Table[Tooltip[{β + ArcSin[(ky Cos[χ] + kx Sin[χ])/(kk Sqrt[1 - (kx Cos[χ] - ky Sin[χ])^2/kk^2])], -θ -
       ArcSin[(kx Cos[χ] - ky Sin[χ])/kk]}, Row[{"ky = ", ky}]], {ky, -0.3, 0.3, 0.03}]],
  {kx, -0.3, 0.3}, PlotStyle → {Black}, AspectRatio → 1, AxesLabel → {ϕ, ψ}]]
Clear[kk, β, θ, χ];
```

Out[230]=



In[232]:=
```
Cos[ArcSin[ϕ]]
```

Out[232]= $\sqrt{1-\phi^2}$